Superconvergence of the Velocity in Mimetic Finite Difference Methods on Quadrilaterals

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Precise calculation of the fluid velocity is important for porous media and other applications. The points or lines where the numerical solution is super-close to the exact solution may be used to improve the accuracy of the overall simulation.

The mimetic finite difference method uses discrete operators that preserve certain critical properties of the original continuum differential operators. Conservation laws, solution symmetries, and the fundamental identities and theorems of vector and tensor calculus are examples of such properties. This "mimetic" technique has been applied successfully to several applications including diffusion [1, 2, 3], magnetic diffusion and electromagnetics [4], continuum mechanics [5], and gas dynamics [6].

A connection between the mimetic finite difference method and the mixed finite element method with Raviart-Thomas finite elements has been established in [7]. In particular, it was shown that the scalar product in the velocity space proposed in [2] for mimetic finite difference methods can be viewed as a quadrature rule in the context of mixed finite element methods.

In [8], we establish superconvergence of the velocity for mimetic finite difference approximations of second-order elliptic problems over h^2-uniform quadrilateral meshes in 2D. This superconvergence result can be applied to obtain superconvergence for the computed velocity at the midpoints of the edges in an h^2 uniform mesh. The superconvergence result holds for a full tensor coefficient. The analysis exploits the relation between mimetic finite differences and mixed finite element methods via a special quadrature rule for computing the scalar product in the velocity space.

Specifically, the main result in [8] is a superconvergence estimate for the velocity in a linear second-order elliptic problem that models single phase Darcy flow, which is usually written as a first-order system for the fluid pressure p and velocity **u**:

$$\begin{aligned} \mathbf{u} &= -\mathbf{K}\nabla p, && \text{in } \boldsymbol{\Omega} \\ \operatorname{div}\mathbf{u} &= f, && \text{in } \boldsymbol{\Omega} \\ \mathbf{u}\,.\,\mathbf{n} &= g, && \text{on } \partial\boldsymbol{\Omega} \end{aligned}$$

where **K** is a symmetric uniformly positive tensor. From this we derive a mimetic finite difference method, where the velocity is computed at centers of mesh edges, and the pressure is computed at centers of cells and edges. Our main result for the accuracy of this finite difference method is summarized as follows.

For the velocity $\mathbf{u}^h$ of the mixed finite difference method, on h^2-uniform quadrilateral grids, there exists a positive constant C independent of h such that

$$|||\mathbf{u} - \mathbf{u}^h||| \leq Ch^2,$$

where $|||\cdot|||$ equation needs to be fixed] is a grid norm that is equivalent to the L^2 norm.

Our numerical results confirm the theoretical superconvergence result. The table shows discretization errors of the flux for a sequence of globally refined grids of the type that is shown in the figure. The convergence rate for the flux that is expected, by just considering standard theory, is one. Our results show that the discrete flux is actually superconvergent, namely of second order.

[1] M. Shashkov and S. Steinberg, "Solving Diffusion Equations with Rough Coefficients in Rough Grids," *J. Comput. Phys.* **129**, 383–405 (1996).
[2] J. M. Hyman, M. Shashkov, and S. Steinberg, "The Numerical Solution of Diffusion Problems in Strongly Heterogeneous Non-isotropic Materials," *J. Comput. Phys.* **132**, 130–148 (1997).

$1/h$	$\vert\vert\vert u - u^h \vert\vert\vert$	$\vert\vert\vert u - u^h \vert\vert\vert$
8	8.32e-2	5.47e-2
16	2.84e-2	1.69e-2
32	8.84e-3	4.49e-3
64	2.42e-3	1.14e-3
128	6.32e-4	2.87e-4
256	1.61e-4	7.17e-5
Rate	1.93	1.99

Table 1—
Convergence rates for the Neumann boundary value problem.

[3] J. E. Morel, R.M. Roberts, and M. Shashkov, "A Local Support-Operators Diffusion Disretization Scheme for Quadrilateral *r–z* Meshes," *J. Comput. Phys.* **144**, 17–51 (1998).

[4] J. M. Hyman and M. Shashkov, "Mimetic Discretizations for Maxwell's Equations and the Equations of Magnetic Diffusion," *Progress in Electromagnetic Research* **32**, 89–121 (2001).

[5] L. Margolin, M. Shashkov, and P. Smolarkiewicz, "A Discrete Operator Calculus for Finite Difference Approximations," *Comput. Meth. Appl. Mech. Engrg.* **187**, 365–383 (2000).

[6] J. Campbell and M. Shashkov, "A Tensor Artificial Viscosity using a Mimetic Finite Difference Algorithm," *J. Comput. Phys.* **172**, 739–765 (2001).

[7] M. Berndt, K. Lipnikov, J.D. Moulton, and M. Shashkov, "Convergence of Mimetic Finite Difference Discretizations of the Diffusion Equation," *J. Numer. Math.* **9**, 253–284 (2001).

[8] M. Berndt, K. Lipnikov, M. Shashkov, M.F. Wheeler, and I. Yotov, "Superconvergence of the Velocity in Mimetic Finite Difference Methods on Quadrilaterals," Los Alamos National Laboratory report LA-UR 03-7904 (2004).

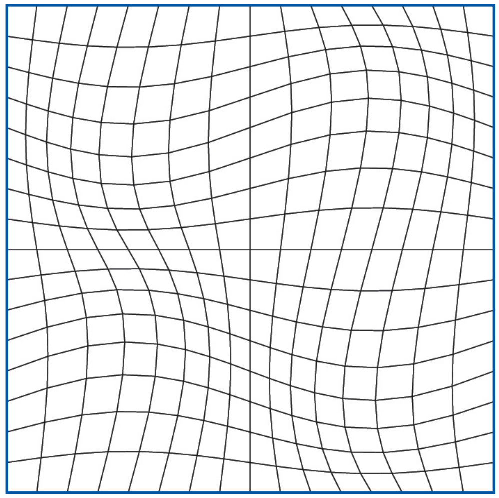

Figure 1—
An example of an h^2 uniform mesh.

**A quadrilateral mesh is called h^2-uniform if any cell and also two neighboring cells combined form a parallelogram, up to an h^2 error.*